

ON GEOMETRIC QUANTIZATION OF THE DIRAC MAGNETIC MONOPOLE

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Abstract. We give a simple derivation of the spectrum of the Dirac magnetic monopole on a unit sphere based on geometric quantization and the Frobenius reciprocity formula.

1. INTRODUCTION

The Dirac magnetic monopole is one of the most remarkable and one of the first integrable systems of quantum mechanics. In his pioneering paper [4] Dirac showed that an isolated magnetic charge q should be quantized: $q \in \mathbb{Z}$. The corresponding Schrödinger equation was solved by Tamm [14] while he was visiting Dirac in Cambridge in 1931.

On the other hand it took quite a while to understand the global nature of the corresponding eigenfunctions. For the Dirac magnetic monopole on a sphere it was done only in 1976 by Wu and Yang [16], who explained that the corresponding wavefunctions (known as *monopole harmonics*) are sections of a complex line bundle L over S^2 and found the spectrum to be

$$(1) \quad \lambda = \left[l(l+1) + |q| \left(l + \frac{1}{2} \right) \right], l = 0, 1, 2, \dots \text{ with degeneracy } 2l + |q| + 1.$$

This gives also a geometric interpretation of Dirac's quantization condition: magnetic charge is the first Chern class $c_1(L) = q$ of the bundle L , which must be an integer.

A different derivation of this result was given by Ferapontov and one of the authors [5], who extended the classical factorisation method going back to Darboux and Schrödinger [13] to curved surfaces. This provided an explicit description of the monopole harmonics by recursive application of the lowering operators to the ground states, which, under the isomorphism $S^2 \cong \mathbb{CP}^1$, are given for positive q by polynomials of degree $\leq q$.

We present here a simple derivation of the spectrum of the Dirac monopole on a unit sphere using geometric quantization. We should say that geometric quantization of the Dirac magnetic monopole and related problems were already discussed in [10, 15], but we believe that our approach is simpler and clearer.

The initial point for us was the calculation by Novikov and Schmelzer [11] of the canonical symplectic structure on the coadjoint orbits of the Euclidean group $E(3)$ of motions of \mathbb{E}^3 , which showed the relation with the classical Dirac monopole (see the next section). A similar calculation for Poincaré and Galilean groups was done by Reiman [12], who also seems to have the idea of geometric quantization in mind, but did not pursue it.

We first show that Novikov-Schmelzer variables have a natural quantum version as covariant derivatives acting on the space of sections $\Gamma(L)$ of the corresponding

line bundle L . Fierz's modification [6] of the angular momentum in the presence of the Dirac magnetic monopole appears naturally in this relation.

The space $\Gamma(L)$ is the representation of $SU(2)$ induced from the representation of $U(1) \subset SU(2)$ given by $z \rightarrow z^q, z \in U(1)$. This space can be decomposed as an $SU(2)$ -module using the classical Frobenius reciprocity formula [7]. We show that the formula for the Dirac monopole spectrum (1) is a simple corollary of this.

2. THE COADJOINT ORBITS OF THE EUCLIDEAN GROUP $E(3)$

Let $e(3)$ be the Lie algebra of the Euclidean group $E(3)$ of motions of \mathbb{E}^3 . It has the basis $l_1, l_2, l_3, p_1, p_2, p_3$, where p and l are generators of translations and rotations (momentum and angular momentum) respectively.

The dual space $e(3)^*$ with the coordinates $\{l_1, l_2, l_3, p_1, p_2, p_3\}$ has the canonical Poisson bracket

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \quad \{l_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0.$$

Its symplectic leaves are the coadjoint orbits of $E(3)$, which are the level sets of the Casimir functions

$$C_1 := (p, p) = R^2, \quad C_2 := (l, p) = \alpha R$$

Following Novikov and Schmelzer [11] introduce

$$(2) \quad \sigma_i = l_i - \frac{\alpha}{R} p_i$$

to identify the coadjoint orbits with T^*S^2 :

$$(p, p) = R^2, \quad (\sigma, p) = 0$$

where TS^2 and T^*S^2 have been identified using the standard Riemannian metric on the radius R sphere.

The new coordinates $\{\sigma_1, \sigma_2, \sigma_3, p_1, p_2, p_3\}$ have Poisson brackets

$$(3) \quad \{\sigma_i, \sigma_j\} = \epsilon_{ijk} \left(\sigma_k - \frac{\alpha}{R} p_k \right), \quad \{\sigma_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0$$

Novikov and Schmelzer computed the canonical symplectic form on the coadjoint orbits and showed that it is given by

$$(4) \quad \omega = dP \wedge dQ + \frac{\alpha}{R^2} dS$$

where $dP \wedge dQ$ is the standard symplectic form on T^*S^2 and dS is the area form on S^2 (see also [12]). As it was pointed out in [11] the second term corresponds to the magnetic field of the (non-quantized) Dirac monopole:

$$\mathcal{H} = \frac{\alpha}{R^2} dS.$$

The value

$$q = \frac{1}{2\pi} \int_{S^2} \mathcal{H}$$

is called the *charge* of the Dirac monopole. Dirac's *quantization condition* [4] is

$$q = \frac{1}{2\pi} \int_{S^2} \mathcal{H} = \frac{1}{2\pi} \int_{S^2} \frac{\alpha}{R^2} dS = 2\alpha \in \mathbf{Z}.$$

Comparing this with (4) we see that this is identical to the *geometric quantization* condition [8]:

$$\frac{1}{2\pi} \int_{[S^2]} \omega = \frac{1}{2\pi} \int_{S^2} \frac{\alpha}{R^2} dS \in \mathbf{Z}.$$

3. LINE BUNDLES OVER S^2 AND QUANTIZATION OF NOVIKOV-SCHMELZER VARIABLES

It is convenient to use the scaled variables

$$(5) \quad x_i = p_i/R, \quad x^2 = x_1^2 + x_2^2 + x_3^2 = 1$$

to work with the unit sphere S^2 .

The quantum version of the Poisson brackets (3) are the following commutation relations (we are using the units in which Planck's constant $\hbar = 1$)

$$(6) \quad [\hat{\sigma}_k, \hat{\sigma}_l] = i\epsilon_{klm}(\hat{\sigma}_m - \alpha\hat{x}_m), \quad [\hat{\sigma}_k, \hat{x}_l] = i\epsilon_{klm}\hat{x}_m, \quad [\hat{x}_k, \hat{x}_l] = 0.$$

We are going to show now that this algebra has a natural representation on the space of sections of a certain line bundle over S^2 .

Recall that a connection on a vector bundle ξ over manifold M^n associates to every vector field X on M^n the operator of covariant derivative ∇_X acting on sections of ξ . The corresponding curvature tensor \mathcal{R} is defined for each pair of vector fields X, Y as

$$\mathcal{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]},$$

where $[X, Y]$ is the standard Lie bracket of vector fields (see e.g. [9]).

Consider a complex line bundle over S^2 with a $U(1)$ -connection having the curvature form

$$\mathcal{R} = i\mathcal{H} = i\alpha dS,$$

which is motivated by geometric quantization. Since the first Chern class of the bundle must be an integer we have

$$q = \frac{1}{2\pi i} \int_{S^2} \mathcal{R} = \frac{1}{2\pi} \int_{S^2} \alpha dS = 2\alpha \in \mathbf{Z},$$

which is precisely Dirac's quantization condition.

Let

$$X_1 = x_3\partial_2 - x_2\partial_3, \quad X_2 = x_1\partial_3 - x_3\partial_1, \quad X_3 = x_2\partial_1 - x_1\partial_2$$

be the vector fields generating rotations of S^2 and ∇_{X_j} be the corresponding covariant derivatives. We claim that

$$\hat{\nabla}_j := i\nabla_{X_j}$$

and the operators \hat{x}_j of multiplication by x_j satisfy the commutation relations (6).

Indeed, by definition of the curvature form, we have

$$\mathcal{R}(X_1, X_2) = \nabla_{X_1} \nabla_{X_2} - \nabla_{X_2} \nabla_{X_1} - \nabla_{[X_1, X_2]} = i\alpha X_3$$

since

$$\alpha dS(X_1, X_2) = \alpha \begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \end{vmatrix} = \alpha x_3(x_1^2 + x_2^2 + x_3^2) = \alpha x_3.$$

This implies

$$[\nabla_{X_1}, \nabla_{X_2}] = \nabla_{X_3} + i\alpha\hat{x}_3$$

since $[X_1, X_2] = X_3$. Similarly we have

$$[\hat{\nabla}_k, \hat{\nabla}_l] = i\epsilon_{klm}(\hat{\nabla}_m - \alpha\hat{x}_m)$$

for all $k, l, m = 1, 2, 3$. The rest of the relations (6) are obvious.

Alternatively, we can look for the quantization of Novikov-Schmelzer variables as covariant derivatives:

$$\hat{\sigma}_j = i\nabla_{X_j}.$$

Then the same calculation shows that the curvature form of the corresponding connection must be $i\alpha dS$.

Finally returning to the original variables we have the operators

$$(7) \quad \hat{l}_j = \hat{\nabla}_j + \alpha x_j,$$

which coincides with the famous modification of the angular momentum in the presence of the Dirac magnetic monopole [6]. This provides us with one more explanation of this well known, but a bit mysterious¹ physical notion.

4. INDUCED REPRESENTATIONS AND FROBENIUS RECIPROCITY

Let L_q be the complex line bundle over S^2 with first Chern class q . We are interested in the space $\Gamma(L_q)$ of sections of L_q . Viewing S^2 as $SU(2)/U(1)$ (with $U(1)$ as the diagonal subgroup) we have a natural interpretation of $\Gamma(L_q)$ as a representation of $SU(2)$.

In representation theory this construction is known as an induced representation (see e.g. [7]). One can use the classical Frobenius reciprocity formula from this theory to decompose $\Gamma(L_q)$ into irreducible $SU(2)$ modules.

First recall that all finite-dimensional irreducible representations of $SU(2)$ are labelled by a highest weight $k \in \mathbf{Z}_{\geq 0}$. The corresponding spaces V_k have dimension $k+1$ and weights

$$(8) \quad -k, -k+2, \dots, k-2, k.$$

All finite-dimensional irreducible representations W_q of $U(1)$ have dimension 1 and are given by

$$e^{i\theta} \mapsto e^{iq\theta}, \quad q \in \mathbf{Z}.$$

One can use W_q to induce a representation $\text{ind}_{U(1)}^{SU(2)}(W_q)$ of $SU(2)$, which can be described geometrically as the space of sections of the line bundle L_q over S^2 with the first Chern class q (see e.g. [2]):

$$\Gamma(L_q) = \text{ind}_{U(1)}^{SU(2)}(W_q).$$

This induced representation is not irreducible. To decompose it we will use the *Frobenius reciprocity formula*

$$(9) \quad \left\langle V, \text{ind}_H^G(W) \right\rangle_G = \left\langle W, \text{res}_G^H(V) \right\rangle_H.$$

Here G is a group, H is its subgroup, V and W are the irreducible representations of G and H respectively, $\text{ind}_H^G(W)$ is the representation of G induced from W , $\text{res}_G^H(V)$ is the restriction of the representation V to the subgroup H and the brackets denote the multiplicity of the first representation entering into the second one (see e.g. [7]).

¹Sidney Coleman, in his famous lectures on Dirac monopoles [3], wrote about this modification of angular momentum: "The second term looks very strange indeed; in Rabi's immortal words about something else altogether, "Who ordered that ?""

In our concrete case we have

$$(10) \quad \left\langle V_k, \text{ind}_{U(1)}^{SU(2)}(W_q) \right\rangle_{SU(2)} = \left\langle W_q, \text{res}_{SU(2)}^{U(1)}(V_k) \right\rangle_{U(1)}.$$

Since the restriction is the sum of the weight spaces

$$\text{res}_{SU(2)}^{U(1)}(V_k) = \bigoplus_{j \in S_k} W_j,$$

where $S_k = \{-k, -k+2, \dots, k-2, k\}$ we see that each V_k , which (after restriction) contains W_q will appear once in the decomposition of $\Gamma(L_q)$ and this can happen only if $k \geq |q|$ and $k - |q|$ is even. Therefore $\Gamma(L_q)$ decomposes into $SU(2)$ -modules as

$$(11) \quad \text{ind}_{U(1)}^{SU(2)}(W_q) = \Gamma(L_q) = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} V_{2l+|q|}.$$

5. CALCULATION OF THE SPECTRUM OF THE DIRAC MONOPOLE

The Hamiltonian of the Dirac monopole can be written in terms of Novikov-Schmelzer operators as

$$H = \hat{\sigma}^2$$

or, equivalently, in terms of magnetic angular momentum \hat{l} as

$$H = \hat{l}^2 - \alpha^2 = \hat{l}^2 - \frac{1}{4}q^2.$$

Since the components of \hat{l}_m satisfy the standard commutation relations

$$[\hat{l}_k, \hat{l}_m] = i\epsilon_{kmn}\hat{l}_n,$$

the operator \hat{l}^2 is a Casimir operator for $SU(2)$ and acts on V_k as a scalar: if $s = k/2$ then

$$(12) \quad \hat{l}^2 = s(s+1) = \frac{1}{4}k(k+2),$$

see e.g. [7]. The space $V_{2l+|q|}$ has dimension $2l + |q| + 1$, and for $\psi \in V_{2l+|q|}$, the operator H acts as

$$H\psi = \left(\hat{l}^2 - \frac{1}{4}q^2\right)\psi = \left[\frac{1}{4}(2l + |q|)(2l + |q| + 2) - \frac{1}{4}q^2\right]\psi = \left[l(l+1) + |q|\left(l + \frac{1}{2}\right)\right]\psi$$

Thus for a Dirac monopole of charge q the spectrum is

$$\left[l(l+1) + |q|\left(l + \frac{1}{2}\right)\right], l \in \mathbb{Z}_{\geq 0} \text{ with degeneracy } 2l + |q| + 1$$

agreeing exactly with (1).

The corresponding eigenfunctions were computed using a version of Darboux-Schrödinger factorisation method in [5], where the ground eigenstates were identified for non-negative q with the space of polynomials of degree $\leq q$. In our picture the ground eigenspace corresponds to the subspace of holomorphic sections of L_q , which by the Borel-Weil theorem [2, 7] can be identified with the corresponding irreducible $SU(2)$ -module V_q .

Finally, we would like to mention that much of this can be generalised for coadjoint orbits of a semisimple Lie group [1].

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